

A COMBINATORIAL PROOF OF AN IDENTITY FOR THE DIVISOR GENERATING FUNCTION

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1. INTRODUCTION

In [4], Uchimura proved the following q -series identity

$$(1.1) \quad \sum_{k \geq 1} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q; q)_k (1 - q^k)} = \sum_{k \geq 1} \frac{q^k}{1 - q^k},$$

where $(a; b)_k = (1 - a)(1 - ab)(1 - ab^2) \cdots (1 - ab^{k-1})$. The formula (1.1) has been known since 100 years ago ([2]). This identity is an infinite version of the following q -series identity called *Problem 6407* in American Mathematical Monthly[5].

$$\sum_{k=1}^m (-1)^{k-1} \begin{bmatrix} m \\ k \end{bmatrix} \frac{q^{\frac{k(k+1)}{2}}}{1 - q^k} = \sum_{k=1}^m \frac{q^k}{1 - q^k}.$$

Here $\begin{bmatrix} m \\ k \end{bmatrix}$ is a q -binomial coefficient. Many authors have generalized these identities (see e.g. [6]). In this paper, we translate these identities and Dilcher's generalization [1] into combinatorics of partitions, and give a combinatorial proof of them. For example, we transform (1.1) as

$$\sum_{\lambda \in \mathcal{SP}} (-1)^{\ell(\lambda)-1} \lambda_{\ell(\lambda)} q^{|\lambda|} = \sum_{n \geq 1} \sigma_0(n) q^n.$$

It is a q -series identity about *strict partitions* and a *divisor function*.

The generalizations of (1.1) we give in this paper are the following.

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^k \frac{b^k q^{\frac{k(k+1)}{2} + (m-1)k}}{(bq; q)_k (1 - q^k)^m} &= \sum_{j_1=1}^{\infty} \frac{b^{j_1} q^{j_1}}{1 - q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1 - q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1 - q^{j_m}}, \\ \sum_{k=1}^t (-1)^k \frac{b^k q^{\frac{k(k+1)}{2} + (m-1)k}}{(1 - q^k)^m} \begin{bmatrix} t \\ k \end{bmatrix}_{q,b} &= \sum_{j_1=1}^t \frac{b^{j_1} q^{j_1}}{1 - q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1 - q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1 - q^{j_m}}. \end{aligned}$$

As a by-product of their proofs, we obtain some combinatorial results.

2. YOUNG DIAGRAMS

Definition 2.1. Let n be a positive integer. A partition λ of n is an integer sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$$

satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. We call $\ell(\lambda) := \ell$ the length of λ , and each λ_i a part of λ . The set of partitions of n is denoted by $\mathcal{P}(n)$.

Definition 2.2. A partition λ is said to be strict if $\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0$. The set of strict partitions of n is denoted by $\mathcal{SP}(n)$.

Definition 2.3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. The Young diagram of λ is defined by

$$Y(\lambda) := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}.$$

We call $(i, j) \in Y(\lambda)$ the (i, j) -cell of λ . And the set of the corners of λ is defined by

$$C(\lambda) := \{(i, j) \in Y(\lambda) \mid (i+1, j), (i, j+1) \notin Y(\lambda)\}.$$

We put $c(\lambda) := \#C(\lambda)$, the number of the corners of λ .

Definition 2.4. Let $(i, j) \in Y(\lambda)$, The (i, j) -hook length of λ is defined by

$$h_{ij}(\lambda) := \#\{(i', j') \in Y(\lambda) \mid i' = i, j' \geq j \text{ or } j' = j, i' \geq i\}$$

And we put $a_{ij}(\lambda) := \lambda_i - j + 1$, the (i, j) -arm length of λ . We remark that our definition of arm length a_{ij} is different by 1 from the usual definition [3].

3. q -SERIES IDENTITY FOR THE DIVISOR FUNCTION

Theorem 3.1. (*Uchimura's identity*)

$$\sum_{k \geq 1} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q; q)_k (1 - q^k)} = \sum_{k \geq 1} \frac{q^k}{1 - q^k},$$

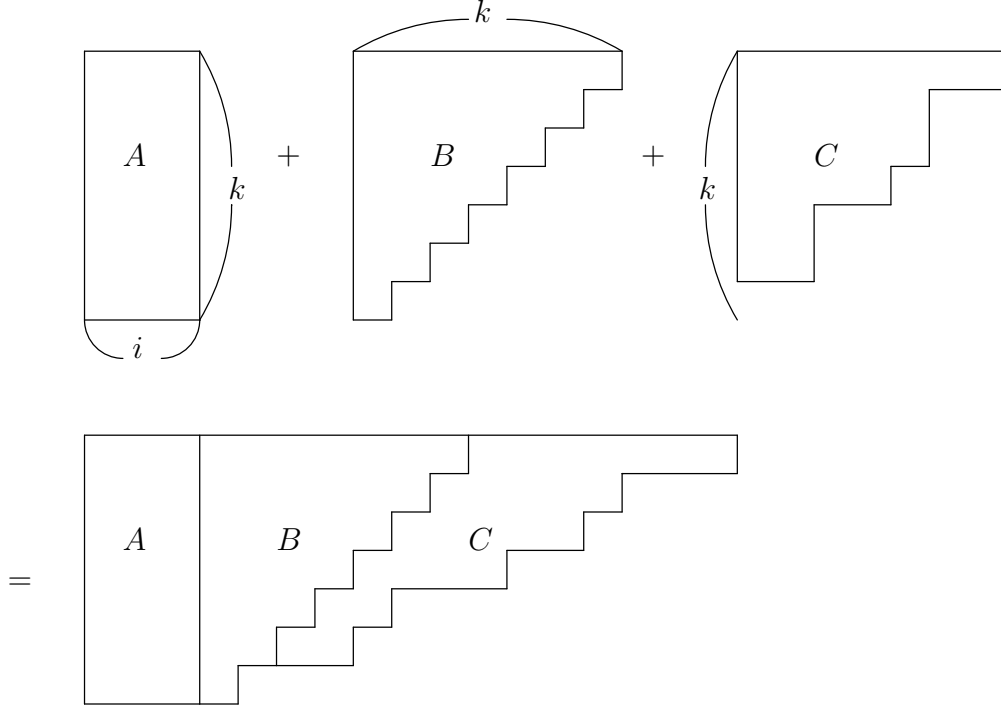
where $(a; b)_k = (1 - a)(1 - ab)(1 - ab^2) \cdots (1 - ab^{k-1})$.

Remark that the right-hand side is computed as

$$\sum_{k \geq 1} \frac{q^k}{1 - q^k} = \sum_{k=1}^{\infty} (q^k + q^{2k} + q^{3k} + \cdots) = \sum_{n \geq 1} \sigma_0(n) q^n,$$

where $\sigma_0(n)$ is the number of positive divisors of n . We now translate this identity into a language of Young diagrams. Then we are able to prove this identity combinatorially.

Figure 1.



Looking at each term of the left-hand side, $q^{\frac{k(k+1)}{2}}$ is translated into the stairs B in Figure 1. Since $\frac{1}{(q; q)_k}$ is the generating function of partitions whose lengths are at most k , this term corresponds to C in Figure 1. The leftover $\frac{1}{1 - q^k}$ is the generating function of rectangular Young diagrams whose vertical lengths are equal to k . This part corresponds to A in Figure 1. Therefore the left-hand side of the identity is an alternating sum over $k \geq 1$ of $A + B + C$. As is noted in Figure 1, the “sum” $A + B + C$ is a strict partition. For a strict partition λ , we count the number of tuples (A, B, C) such that $A + B + C = \lambda$. Let λ be a fixed strict partition of length k . One can embed the stairs B into λ in λ_k ways. For each embedding the rectangle A and the partition C are uniquely determined, respectively. Therefore there are λ_k tuples (A, B, C) , such that $A + B + C = \lambda$. Summing up over k , Theorem 3.1 reads

$$(3.1) \quad \sum_{\lambda \in \mathcal{SP}} (-1)^{\ell(\lambda)-1} \lambda_{\ell(\lambda)} q^{|\lambda|} = \sum_{n \geq 1} \sigma_0(n) q^n.$$

Theorem 3.2. For any positive integers n and k ,

$$\begin{aligned} & \# \{ \lambda \in \mathcal{SP}(n) \mid \lambda_1 \geq k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda) : \text{odd} \} \\ & - \# \{ \lambda \in \mathcal{SP}(n) \mid \lambda_1 \geq k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda) : \text{even} \} \\ & = \begin{cases} 1 & (k \mid n) \\ 0 & (k \nmid n). \end{cases} \end{aligned}$$

Example. For $n = 5$, we draw Young diagrams $Y(\lambda)$ of all strict partitions of 5, and write arm length in $(1, j)$ -cell for $1 \leq j \leq \lambda_{\ell(\lambda)}$.

Figure 2.

$$\begin{array}{|c|c|c|c|c|} \hline 5 & 4 & 3 & 2 & 1 \\ \hline \end{array} \quad - \quad \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline & & & \\ \hline \end{array} \quad - \quad \begin{array}{|c|c|c|} \hline 3 & 2 & \\ \hline & & \\ \hline \end{array}$$

$$= \quad 5, 1.$$

Here the numbers are regarded as variables. Number 5 and 1 are the positive divisors of 5.

Proof of Theorem 3.2

We consider the set of strict partitions of n such that $a_{1,j} = k$ for $1 \leq j \leq \lambda_{\ell(\lambda)}$:

$$\mathfrak{D}(n, k) := \{ \lambda \in \mathcal{SP}(n) \mid \lambda_1 - \lambda_{\ell(\lambda)} < k \leq \lambda_1 \}.$$

We divide these strict partitions into two classes A and B :

$$A = \{ \lambda \in \mathfrak{D}(n, k) \mid \forall i, k \nmid \lambda_i \}, \quad B = \{ \lambda \in \mathfrak{D}(n, k) \mid \exists i, k \mid \lambda_i \}.$$

We consider a map between them that changes the length by 1.

$$\alpha_k : A \rightarrow B, \quad \alpha_k(\lambda) = \lambda',$$

where $\lambda' \in B$ is defined in the following steps:

Step 1. Append 0 in the tail of λ to get $(\lambda_1, \dots, \lambda_{\ell+1})$.

Step 2. Subtract k from $\max(\lambda_1, \dots, \lambda_{\ell+1})$, and add k to $\lambda_{\ell+1}$.

Step 3. Repeat Step 2 till $\max(\lambda_1, \dots, \lambda_{\ell+1}) - \min(\lambda_1, \dots, \lambda_{\ell+1})$ gets less than k .

Step 4. From the resulting composition we have the partition $\lambda' = (\lambda_1, \dots, \lambda_{\ell+1})$, by arranging parts.

By the above construction, we have $\ell(\lambda') = \ell(\lambda) + 1$, and

$$\# \{ i \mid \lambda_i \pmod{k} \equiv j, 1 \leq i \leq \ell \} = \# \{ i \mid \lambda'_i \pmod{k} \equiv j, 1 \leq i \leq \ell \}$$

for $1 \leq j \leq k-1$. The partition that α_k can not pair up is $(\lambda_1, \dots, \lambda_{\ell}) \equiv (0) \pmod{k}$. Therefore $\lambda = (n)$ is left when n is a multiple of k . \square

Proof of theorem 3.1

Sum of the left-hand side of Theorem 3.2 over k is

$$\begin{aligned} & \sum_{k \geq 1} \#\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geq k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda) : \text{odd}\} \\ & - \sum_{k \geq 1} \#\{\lambda \in \mathcal{SP}(n) \mid \lambda_1 \geq k > \lambda_1 - \lambda_{\ell(\lambda)}, \ell(\lambda) : \text{even}\} \\ & = \sum_{\lambda \in \mathcal{SP}(n)} (-1)^{\ell(\lambda)-1} \lambda_{\ell(\lambda)}. \end{aligned}$$

And sum of the right-hand side is

$$\sum_{k|n} 1 = \sigma_0(n).$$

They are the coefficients of q^n in (3.1). \square

Theorem 3.1 is the generating function for the total sum of Theorem 3.2. Taking the sum over k from 1 to m for Theorem 3.1, we have the identity of Problem 6407 which is shown in the following generating function.

$$\sum_{k=1}^m (-1)^{k-1} \begin{bmatrix} m \\ k \end{bmatrix} \frac{q^{\frac{k(k+1)}{2}}}{1 - q^k} = \sum_{k=1}^m \frac{q^k}{1 - q^k},$$

where the q -binomial coefficient is defined by

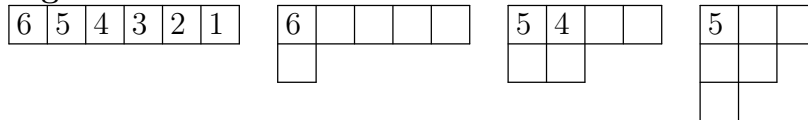
$$\begin{bmatrix} m \\ k \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} & 1 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.3. For $n = 2(2m + 1)$,

$$\begin{aligned} & \sum_{\lambda \in \mathcal{SP}(n)} \#\{h_{1,i}(\lambda) \mid i \leq \lambda_{\ell(\lambda)}, \text{ odd}\} \\ & = \sum_{\lambda \in \mathcal{SP}(n)} \#\{h_{1,i}(\lambda) \mid i \leq \lambda_{\ell(\lambda)}, \text{ even}\}. \end{aligned}$$

Example. For $n = 6$, we draw Young diagrams $Y(\lambda)$ of all strict partitions of 6, and write hook lengths in $(1, j)$ -cell for $1 \leq j \leq \lambda_{\ell(\lambda)}$.

Figure 3.



The number of odd numbers equals the number of even numbers.

Proof. In Theorem 3.2, the strict partitions they have same arm length and different parity length are pair. Recall that $h_{1j}(\lambda) = a_{1j}(\lambda) + \ell(\lambda) - 1$. Therefore the parity of their hook length are different. And the leftovers are divisors of n . When n equals $2(2m + 1)$, the number of odd divisors of n equals the number of even divisors of n . \square

4. GENERALIZATIONS

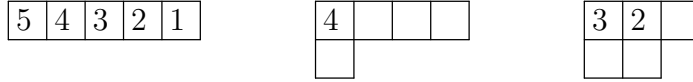
Theorem 4.1. For any positive integers k, m, n ,

$$\begin{aligned} & \# \{ (\lambda, i_1, \dots, i_m) \mid \lambda \in \mathcal{SP}(n), 1 \leq i_1 < \dots < i_m \leq \lambda_{\ell(\lambda)}, a_{1, i_m}(\lambda) = k, \ell(\lambda) \text{ is odd} \} \\ - & \# \{ (\lambda, i_1, \dots, i_m) \mid \lambda \in \mathcal{SP}(n), 1 \leq i_1 < \dots < i_m \leq \lambda_{\ell(\lambda)}, a_{1, i_m}(\lambda) = k, \ell(\lambda) \text{ is even} \} \\ = & \# \{ (\lambda, t_1, \dots, t_{m-c(\lambda)}) \mid \lambda \in \mathcal{P}(n), 1 \leq t_1 < \dots < t_{m-c(\lambda)} < \ell(\lambda), \lambda_1 = k, \lambda_{t_i} = \lambda_{t_i+1} \}. \end{aligned}$$

When $m = 1$, this identity is equivalent to Theorem 3.2.

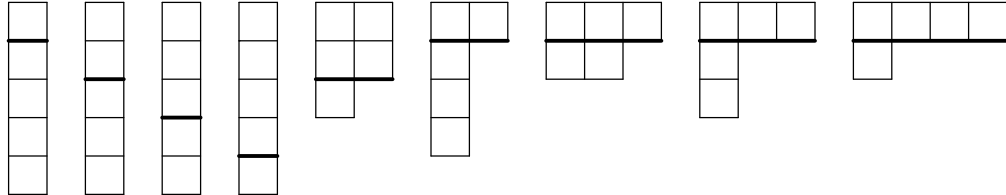
Example. @For $n = 5, m = 2$, we draw the same figure as Figure 2.

Figure 4.



We count the pairs of arm lengths in the same partition λ with sign $(-1)^{\ell(\lambda)}$. The pairs with positive sign are $(5, 1)$, $(4, 1)$, $(3, 1)$, $(2, 1)$, $(5, 2)$, $(4, 2)$, $(3, 2)$, $(5, 3)$, $(4, 3)$, $(5, 4)$. The pair with negative sign is $(3, 2)$. On the other hand, the Young diagrams of size 5 they are made by concatenating 2 rectangles are,

Figure 5.



There are 9 such Young diagrams. Here we count the ways of concatenating rectangles. Hence, for example, the first 4 diagrams must be

thought of as the different ones. Therefore the number of pairs with positive sign minus the number of pairs with negative sign equals the number of Young diagrams made by concatenating 2 rectangles. And more fix smaller number b , the number of pair (a, b) with positive sign minus the number of pair (a, b) with negative sign also equals the number of Young diagrams λ made by concatenating 2 rectangles that λ_1 equals b . For example, the number of the pairs that smaller number is 1 and the number of Young diagrams that size of first line is 1 are both 4.

Proof. When $\lambda \in \mathcal{SP}(n)$, $i_1 < \dots i_m \leq \lambda_\ell$ are given, we make strict partitions $\lambda^{(1)}, \dots, \lambda^{(m)}$ in the following procedure. First, we put $\lambda^{(1)}$ with λ . When $\lambda^{(h)}$ is determined, we put $\mu = \lambda^{(h)}$ and $j_h = a_{1, i_{m+1-h}}(\lambda^{(h)})$. And we make new strict partition $[\mu]$ by replacing the maximum parts μ_1 by $\mu_1 - j_h$. This operation keeps strictness, since μ is strict in mod k . We repeat this operation until we get $a_{1, i_{m-h}}(\mu) \leq j_h$. We put $\lambda^{(h+1)}$ with μ obtained in this way. And let t_h be the number of times of the operations. The lengths of $\lambda^{(i)}$ -s are same as the length of λ . We consider that j_m is k of theorem 3.2, and pair up $\lambda^{(m)}$ -s by α_{j_m} . Then there is only 1 difference between the lengths of partitions of each pair. Leftovers are $\lambda^{(m)} = (\lambda_1^{(m)})$ that $\lambda_1^{(m)}$ is multiple of j_m . Since $j_1 \geq \dots \geq j_m$, it corresponds with Young diagram that is made by concatenating rectangles $j_h \times t_h$, where $t_m = \frac{\lambda_1^{(m)}}{j_m}$. \square

Taking the total sum over k for Theorem 4.1, we have an identity which is shown in the following generating function.

Theorem 4.2. (*Dilcher's identity 1*) For any positive integer m ,

$$\sum_{k=1}^{\infty} (-1)^k \frac{q^{\frac{k(k+1)}{2} + (m-1)k}}{(q; q)_k (1 - q^k)^m} = \sum_{j_1=1}^{\infty} \frac{q^{j_1}}{1 - q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1 - q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1 - q^{j_m}}.$$

In a language of Young diagrams, this identity is equivalent to the following.

$$\sum_{\lambda \in \mathcal{SP}} (-1)^{\ell(\lambda)-1} \binom{\lambda_{\ell(\lambda)}}{m} q^{|\lambda|} = \sum_{\lambda \in \mathcal{P}} \binom{\ell(\lambda) - c(\lambda)}{m - c(\lambda)} q^{|\lambda|}.$$

And taking the sum over k from 1 to k for Theorem 4.1, we have an identity which is shown in the following generating function.

Theorem 4.3. (*Dilcher's identity 2*) For any positive integers m and t ,

$$\sum_{k=1}^t (-1)^k \frac{q^{\frac{k(k+1)}{2} + (m-1)k}}{(1-q^k)^m} \begin{bmatrix} t \\ k \end{bmatrix} = \sum_{j_1=1}^t \frac{q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1-q^{j_m}}.$$

Theorem 4.1 is a generalization of Dilcher's identities. Which is the coefficient of $b^k q^n$ in the following.

Theorem 4.4. For any positive integer m ,

$$\sum_{k=1}^{\infty} (-1)^k \frac{b^k q^{\frac{k(k+1)}{2} + (m-1)k}}{(bq; q)_k (1-q^k)^m} = \sum_{j_1=1}^{\infty} \frac{b^{j_1} q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1-q^{j_m}}.$$

Theorem 4.5. For any positive integers m and t ,

$$\sum_{k=1}^t (-1)^k \frac{b^k q^{\frac{k(k+1)}{2} + (m-1)k}}{(1-q^k)^m} \begin{bmatrix} t \\ k \end{bmatrix}_{q,b} = \sum_{j_1=1}^t \frac{b^{j_1} q^{j_1}}{1-q^{j_1}} \sum_{j_2=1}^{j_1} \frac{q^{j_2}}{1-q^{j_2}} \cdots \sum_{j_m=1}^{j_{m-1}} \frac{q^{j_m}}{1-q^{j_m}},$$

where $\begin{bmatrix} t \\ k \end{bmatrix}_{q,b}$ is defined as

$$\begin{bmatrix} t \\ k \end{bmatrix}_{q,b} = \sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 \leq t-k, \ell(\lambda) \leq k}} b^{\lambda_1} q^{|\lambda|}.$$

When $b = 1$, they are Dilcher's identities.

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